# CHSH Game Using Convex Geometry

Jordan McCarney
Supervisor: Dr. Mark Howard

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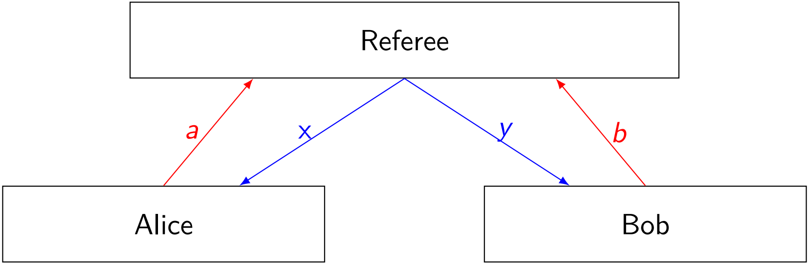
# CHSH Game




Alice and Bob are given inputs $x, y \in \{0, 1\}$ respectively, and their goal is to provide outputs $a, b \in \{0, 1\}$ to satisfy the relation $xy = a \oplus b$.

**Constraint:** No Communication

# CHSH Game - Classical Physics

| x | y | $x \cdot y$ | $a \oplus b$ | Result |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | Win |
| 0 | 1 | 0 | 0 | Win |
| 1 | 0 | 0 | 0 | Win |
| 1 | 1 | 1 | 0 | Lose |

Win rate of 3/4 when a=b

Alice and Bob can win with probability 75% using the strategy of always choosing both $a$ and $b = 0$ or both $a$ and $b = 1$.

75% is the **maximum** success rate that they can achieve classically.

# CHSH Game - Classical Physics

Classical strategies can be deterministic or randomized. For a deterministic strategy, Alice's output $a$ must be a function of her random input $x$.

Therefore, Alice must choose $a(x) = 0$, $a(x) = 1$, $a(x) = x$ or $a(x) = \neg x$. Similarly, Bob must choose $b(y) = 0$, $b(y) = 1$, $b(y) = y$ or $b(y) = \neg y$.

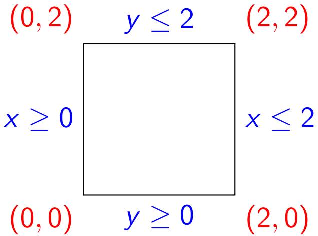
| | $a = 0$ | $a = 1$ | $a = x$ | $a = \neg x$ |
|---|---|---|---|---|
| $b = 0$ | $\frac{3}{4}$ | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{1}{4}$ |
| $b = 1$ | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{1}{4}$ | $\frac{3}{4}$ |
| $b = y$ | $\frac{3}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{3}{4}$ |
| $b = \neg y$ | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{1}{4}$ |

Win Rates of all Deterministic Classical Strategies

# Convex Geometry - What is a Polytope?

A convex polytope is the convex hull of some finite set of points. A given polytope can be defined in two ways, either by its vertex representation or by its half-space representation.

For example, take a square with vertices $\{(0,0),(0,2),(2,0),(2,2)\}$, its half-space representation is then given by the following list of inequalities: $\{x \geq 0,\ x \leq 2,\ y \geq 0,\ y \leq 2\}$



# The Local CHSH Polytope

The vertices for the local CHSH polytope come from the conditional probabilities $P(ab|xy) = P_A(a|x) \cdot P_B(b|y)$ for all $\{a, b, x, y\} \in \{0, 1\}$.

$$\begin{bmatrix} P(00|00) \\ P(00|01) \\ \vdots \\ P(11|11) \end{bmatrix} = \begin{bmatrix} P_A(0|0) \cdot P_B(0|0) \\ P_A(0|0) \cdot P_B(0|1) \\ \vdots \\ P_A(1|1) \cdot P_B(1|1) \end{bmatrix}$$

To produce all the vertices, we must look at all 16 potential scenarios. E.g. Scenario 1 could be to take $P_A(0|0) = 0$, $P_A(0|1) = 0$, $P_B(0|0) = 0$ and $P_B(0|1) = 0$. *Note:* $P_A(0|0) = 0$ implies $P_A(1|0) = 1$ etc.

# The Local CHSH Polytope

Half-space enumeration was performed on the vertices to return 16 inequalities of the form

$$0 \leq \theta + x_1 + \ldots + x_{16}$$

which can then be translated back into inequalites in terms of the conditional probabilities

$$0 \leq \theta + P(00|00) + \ldots + P(11|11)$$

Bell CHSH inequality (local hidden variables)

$$\begin{gathered} P(00|00) - P(01|00) - P(10|00) + P(11|00) \\ +P(00|01) - P(01|01) - P(10|01) + P(11|01) \\ +P(00|10) - P(01|10) - P(10|10) + P(11|10) \\ -P(00|11) + P(01|11) + P(10|11) - P(11|11) \leq 2 \end{gathered}$$

# The Local CHSH Polytope

Looking at scenario 1 from before

$$P_A(0|0) = 0, P_B(0|0) = 0$$
$$P_A(1|0) = 1, P_B(1|0) = 1$$
$$P_A(0|1) = 0, P_B(0|1) = 0$$
$$P_A(1|1) = 1, P_B(1|1) = 1$$

any probability with a 0 in the LHS of the bracket $= 0$, so we are left with the Bell inequality:

$$P(11|00) + P(11|01) + P(11|10) - P(11|11) \leq 2$$

And the four above conditional probabilities $= 1$, so we can see that the CHSH game is satisfied by local hidden-variable models.

# CHSH Game - Quantum Physics

Alice and Bob can beat the classical strategy using quantum mechanics. If Alice and Bob share a qubit each of an EPR pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, they can exploit correlations of the entangled qubits to win with probability $\approx 0.85$

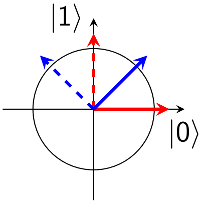
They can do this by choosing the bases that they make their measurement in (producing their outputs $a$ and $b$), depending on the inputs $x$ and $y$ that they receive.

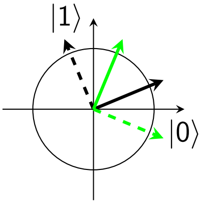
# CHSH - Quantum Physics

$xy = a \oplus b$

**Inputs:**
**Alice, *x*:** $0 \rightarrow$ Blue, $1 \rightarrow$ Red
**Bob, *y*:** $0 \rightarrow$ Black, $1 \rightarrow$ Green

**Outputs *a* and *b*:** Solid $\rightarrow 0$, Dotted $\rightarrow 1$




**Left:** Alice's Bases **Right:** Bob's Bases

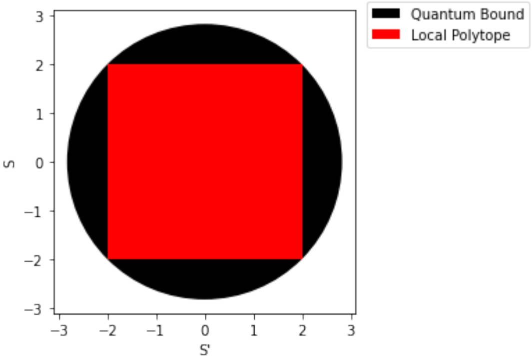
# CHSH - Quantum Physics

Example Scenario: Say Alice and Bob receive $x = y = 0$. If Alice then measures an output of 0, because of the correlation between the EPR pair, Bob's qubit 'snaps' to the solid blue line. To satisfy the relation $xy = a \oplus b$, Bob wants to output the solid black line, which is $\frac{\pi}{8}$ radians away from the solid blue line. Thus his probability of doing so is $cos^2(\frac{\pi}{8}) \approx 0.85$

So regardless of what Alice and Bob receive as their inputs, the output they must measure for success is always is $22.5^\circ$ or $\frac{\pi}{8}$ radians away. Thus the probability for success using quantum physics is $cos^2(\frac{\pi}{8}) \approx 0.85 \geq 0.75$

Maximum value of the Bell inequality from before using quantum mechanics is $P(00|00) + P(00|01) + ... - P(11|11) = 2\sqrt{2} > 2$.

# Violation of Bell CHSH Inequality




Quantum Violation of Bell CHSH Polytope

As can be seen from the figure, and from our analysis of the CHSH game, quantum mechanics allows us to violate Bell inequalities.