Primitive Matrices and Graphs

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1 Introduction

Primitive matrices are a class of matrix that have many uses in both the mathematical world, and the real world. Their main application comes in the form of the Perron-Frobenius theorem, which has sparked many advancements in the worlds of probability theory, economics and social networking.

My goal for this summer project was to explore primitive matrices and their connections with their graph theory equivalents, primitive graphs. I investigated different properties of these primitive matrices and graphs over a finite field $\mathbb{F}_2$, and attempted to find some new and exciting connections between the two subjects.

2 Primitivity

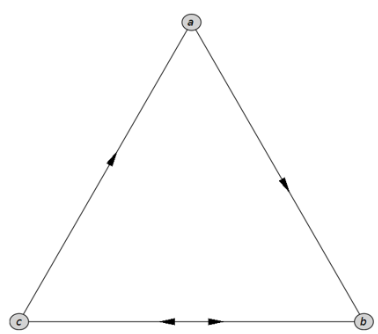
Definition: Primitive matrices are $n \times n$ non-negative matrices A such that $A^k > 0$ for some k. Some examples of primitive matrices are shown below:

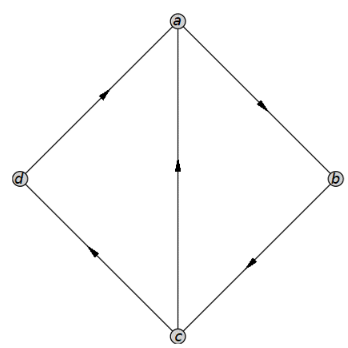
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In the examples above, $A^5 > 0$ and $B^{10} > 0$, which highlights their primitivity. The least such k for which $A^k > 0$ is known as the **exponent** of the matrix. Note also that $A^l > 0$ for $l \geq k$.

Remark: The patterns of positive entries in A, A^2, ... , A^l depends only on the pattern of positive entries in A, and not on their values.

Definition: A primitive directed graph $\Gamma(A)$, on the other hand, is a graph by which there is some k for which there is a walk of length k from x_i to x_j for every pair of vertices x_i, x_j.

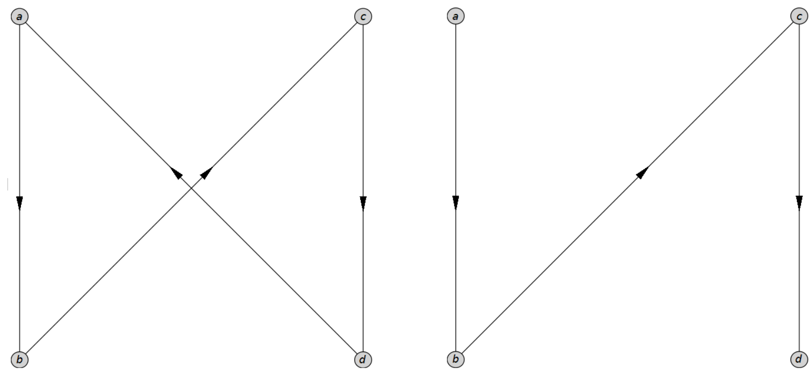




Like their counterparts, the least such k for which there is a walk of length k from x_i to x_j $\forall\, x_i, x_j \in \Gamma(A)$ is known as the **exponent** of the graph. The graphs above are actually generated from the matrices A and B above. This is done by assigning each vertex to a row/column in the matrix. For example, the graph of matrix A, which is a 3×3 matrix, has vertices x_1, x_2 and x_3, with an arc from x_i to x_j if $A_{ij} > 0$. These matrices are known as **adjacency matrices** of the directed graphs.

Remark: $(A^k)_{ij} > 0$ if and only if there is a walk of length k from x_i to x_j in $\Gamma(A)$. Thus as $A^5 > 0$, there is also a walk of length 5 from any vertex in the triangular-shaped graph above to any vertex. This vertex can also be the same i.e. there is a walk of length 5 from vertex a back to vertex a.

Another key concept in this report is the notion of strongly connected graphs and minimal primitivity. A directed graph $\Gamma(A)$ is **strongly connected** if for any pair of vertices x_i and x_j, there is a directed walk between them. For example, the graph below on the left is strongly connected, but the one on the right is not.

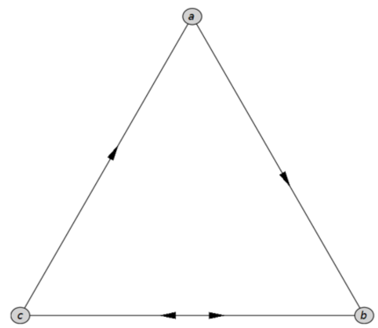


This is due to the fact that there is no directed walk from vertex d to any other vertex, which stops many of the other directed walks between vertices seen in the directed graph on the left.

Definition: A graph Γ is minimally primitive if it is primitive, but removing any arc leaves an imprimitive graph.

There are two ways you can tell easily whether a graph is minimally primitive, the first being that the graph must be strongly connected. A primitive directed graph must always be strongly connected, as otherwise there would not be a directed walk connecting each vertex. Also, for a graph to be primitive, the greatest common divisor of its cycle lengths should be 1. Thus if a directed graph Γ is minimally primitive, removing an arc should result in a case whereby $gcd(\text{Cycle Lengths of } \Gamma) > 1$.

Let us revisit one of the graphs above, the directed graph $\Gamma(A)$ formed from the matrix A.



This graph is minimally primitive, as removing any arc from this graph leaves a graph that is not strongly connected, the one exception to this being the graph formed by removing the arc $c \to b$. In this case, however, we are left with a singular cycle of length 3, so $gcd(\text{Cycle Lengths of }\Gamma) > 1$. This makes the graph imprimitive.

3 Exponent Bounds

Theorem: Let Γ be a minimally primitive graph of order $n \geq 3$ that has no loops, and is not minimally strongly connected. Then:

$$exp(\Gamma) \geq 5$$

The proof for this statement is a long and tedious constructive proof, which would take up most of this report if included. Instead, I will include an outline of how to construct this proof yourself:

Firstly, to avoid trivial cases, loops from a vertex onto itself are not being considered. This rules out the case that $exp(\Gamma) = 1$, as there would be no path from a vertex back to itself.

You must then consider the cases where $exp(\Gamma) = 2$, $exp(\Gamma) = 3$ and $exp(\Gamma) = 4$, by constructing graphs that contain a **critical arc**. This is an arc such that deleting it doesn't disconnect Γ, but leaves an imprimitive graph. Through adding vertices onto these graphs and ruling out all possibilities of primitive graphs, you come to the conclusion that $exp(\Gamma) \geq 5$

There is many other theorems on exponents of primitive graphs and matrices found in [1], which outline different gaps in the exponents of matrices of order n, which proved helpful in forming general minimally primitive digraphs of order $n = 3$ and $n = 4$.

Theorem (Dulmage and Mendelsohn): There is no primitive matrix of odd order n such that

$$n^2 - 3n + 5 \leq exp(A) \leq (n-1)^2 - 1$$

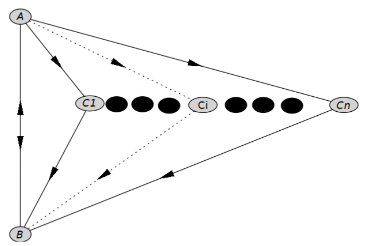
or

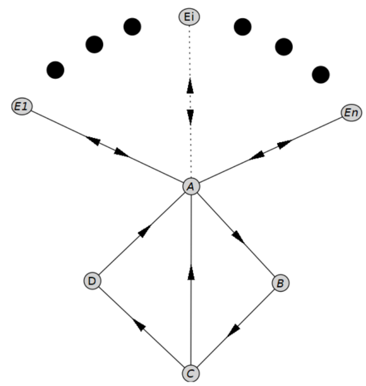
$$n^2 - 4n + 7 \leq exp(A) \leq n^2 - 3n + 1$$

Also, if n is even, then there is no primitive matrix A such that

$$n^2 - 4n + 7 \leq exp(A) \leq (n-1)^2 - 1$$

These theorems were extremely helpful in formulating general minimally primitive digraphs with exponent 5 and 6 respectively.





The graph on the left is a general minimally primitive graph of exponent 5 for a graph Γ_n with $n \geq 3$. To start, we have the graph Γ_3 with vertices A, B, and C_1. This graph is minimally primitive with exponent 5. However, if you keep adding vertices C_i as seen above, the number of vertices increases but the exponent of the graph is still 5. This is because the greatest common divisor of the cycle lengths is 1, but removing any arc leaves a graph that is not strongly connected, or a graph where $gcd(\text{Cycle Lengths of } \Gamma) > 1$. The same can be said about the graph on the right G_n, except this minimally primitive digraph is of exponent 6 with $n \geq 5$.

4 Primitive Matrices and Graphs over $\mathbb{F}_2$

4.1 The Theory

For this next section of my project, I decided to take new angle and view primitivity through the lens of finite field theory. I decided to use $\mathbf{F_2}$ as my finite field of choice as it is one of the more simple fields that can give me an understanding of if there is a connection between primitivity and fields.

To start, we must first define a field. A field is a set of elements $\mathbf{F}$ together with two binary operations, addition and multiplication. Each field must have an additive and multiplicative "identity" element, and must satisfy the following axioms:

- **Associativity:** $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- **Commutativity:** $a + b = b + a$ and $a \cdot b = b \cdot a$
- **Distributivity:** $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
- **Additive Inverses:** $\forall\, a \in \mathbf{F}, \exists - a \in \mathbf{F}$ such that $a + (-a) = 0_{\mathbf{F}}$,
- **Multiplicative Inverses:** $\forall\, a \neq 0 \in \mathbf{F}, \exists\; a^{-1} \in \mathbf{F}$ such that $a \cdot a^{-1} = 1_{\mathbf{F}}$

An example of a field (and the field we will be using) is $\mathbf{F_2}$, where the field only contains the elements 0 and 1. Below are the addition and multiplication tables that describe this field.

$+$	0	1
0	0	1
1	1	0

$\times$	0	1
0	0	0
1	0	1

Let us now pose the question I studied regarding these finite fields:

Question: Let $A \in M_n(\mathbf{F_2})$ be an n x n matrix with entries 0 or 1. Consider this matrix as the adjacency matrix of a directed graph Γ. If $A^k = J$ (with J being the n x n matrix consisting of all 1's) $\forall k \geq m$, what can be said about Γ?

In terms of the graph Γ, this means that there is an integer m with the property that for every pair of vertices u, v in Γ, and for every $k \geq m$, the number of walks of length k from u to v in Γ is odd.

There is a number of arguments about A that we can prove from the start to help us answer the question above:

- $m \leq n$**:** Let $T : \mathbf{F_2^{(n)}} \to \mathbf{F_2^{(n)}}$ be the linear transformation represented by A. If we look at the sequences of images $\mathrm{Im}(T)$, $\mathrm{Im}(T^2)$, ... , these are a sequence of subspaces of $\mathbf{F_2^{(n)}}$ such that $\mathrm{Im}(T) \supseteq \mathrm{Im}(T^2) \supseteq \mathrm{Im}(T^3) \supseteq ...$.
- $A^n = J$ has rank 1. Its columnspace is spanned by the vector j (the n x 1 column vector consisting of only 1's in its entries). j is an eigenvector of A corresponding to the eigenvalue 1. From the Rank-Nullity Theorem, we also know that the the right nullspace of $A^n = J$ has dimension n-1, the kernel of T^n.

Let $\mathcal{B} = \{j, b_2, b_3, ..., b_n\}$ be a basis of $\mathbf{F_2^{(n)}}$ such that $\{b_2, b_3, ..., b_n\}$ is a basis of the nullspace, N. The matrix of T with respect to $\mathcal{B}$ is:

$$A = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & | & & | \\ \vdots & T(b_2) & ... & T(b_n) \\ 0 & | & & | \end{bmatrix}$$

Note: Each $T(b_i)$ belongs to the span of $< b_2, ..., b_n >$. Also, the lower right $(n-1) \times (n-1)$ matrix, Q, is **nilpotent**. This means that $Q^k = 0, \exists\, k \in \mathbf{Z}^+$.

Thus, the basis $\{v_2, ..., v_n\}$ of N can be adjusted to one where Q is a zero matrix with some 1's on the superdiagonal of the matrix. From this, we can assume that A is similar to a matrix of the form (using a 5 x 5 matrix over $\mathbf{F_2}$ as an example):

$$A' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that $*$ can be either a 0 or 1. A'^n is not actually J, as it represents the linear transformation T with respect to the basis $\mathcal{B}$. It is actually a matrix similar to J, shown below:

$$A' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For this example, we need to find a change of basis matrix P to highlight that $A^5 = J$ in the standard basis. To change a matrix from a basis to the standard basis, we use the formula $A = P^{-1}AP$. This change of basis vector P, was found to look like this:

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & & & & \\ 1 & & S & & \\ 1 & & & & \\ 1 & & & & \end{bmatrix}$$

with S being a 4 x 4 matrix with a combination of 0's or 1's as its entries. In my project, I exclusively looked at 5 x 5 matrices over $\mathbf{F_2}$ to get a good grasp of what the connection was between these matrices and the graphs they generated.

4.2 Isomorphism Classes

Definition: An **isomorphism** is a structure preserving mapping between two structures of the same type. An **isomorphism class** is a collection of objects that are isomorphic to eachother.

For this part of the project, I decided to find all of the isomorphism classes for the primitive 5 x 5 matrices A' over $\mathbf{F_2}$ to see if there was any connections between the graphs. This was done by investigating all the different possibilities of A formed by taking all possible A' (of which there are 7, and changing their bases with different change of basis matrices P to see if the resulting matrix is primitive, and if so what its exponent is. I used Maple to do so, with an excerpt of the code below:

```
> with(LinearAlgebra) :
> A := Matrix([
  [1, 0, 0, 0, 0],
  [0, 0, 0, 0, 0],
  [0, 0, 0, 0, 0],
  [0, 0, 0, 0, 0],
  [0, 0, 0, 0, 0]
  ]);
```

$$A := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

> A^2 **mod** 2;

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> P := Matrix([
  [1, 1, 1, 1, 1],
  [1, 1, 0, 0, 0],
  [1, 0, 1, 0, 0],
  [1, 0, 0, 1, 0],
  [1, 0, 0, 0, 1]
  ]);
```

$$P := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

I first added the matrices A' and P into the program. One thing to note is that P had to have an even number of 1's in each row to avoid being a singular matrix, (i.e. a matrix who's determinant is zero and doesn't have an inverse)

$A1 := Multiply(Multiply(MatrixInverse(P), A), P) \ \mathbf{mod}\ 2;$

$$A1 := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$A1^2 \ \mathbf{mod}\ 2;$

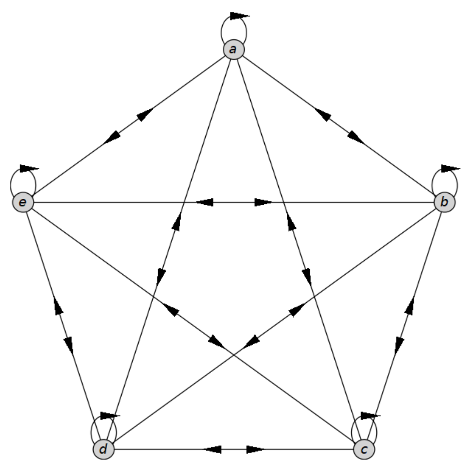
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

I then performed the change of basis operation on A', which in this case resulted in the trivial J matrix. In this case, the matrix is most definitely primitive, with an exponent of 1.

```
with(GraphTheory) :
A := Digraph([a, b, c, d, e], {[a, a], [a, b], [a, c], [a, d], [a, e], [b, a], [b, b], [b, c], [b, d], [b, e], [c, a], [c, b], [c, c], [c, d], [c, e], [d, a], [d, b], [d, c], [d, d], [d, e],
    [e, a], [e, b], [e, c], [e, d], [e, e], })
```

A := Graph 1: a directed graph with 5 vertices, 20 arc(s), and 5 self-loop(s)

```
DrawGraph(A)
```



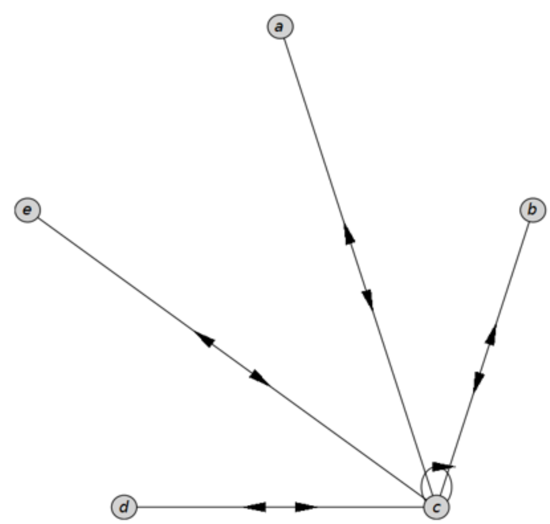
I also used Maple to construct the graphs I have been using throughout this report, and as we can see here, A has constructed the fully connected graph K_5. This is a graph where every vertex is connected to every vertex by an arc, including an arc to itself.

I then conducted this using a variety of change of basis matrices and options for A', and from this found a total of 28 isomorphism classes. Below I will showcase some interesting facts and an example from these classes:

- The graphs ranged from 7 arcs to 25 arcs, with only an odd number of arcs being used in them.

- The graphs always had either 1, 3, or 5 self-loops (i.e. an arc from a vertex to itself).
- Below is an interesting example with only 9 arcs:

$$A := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad P := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad A1 := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$



In this example, $A1^2 = J$, and thus the exponent of this graph is 2. This means that for any $k > 2$, there is an odd number of walks of length k from vertices x_i to x_j, $\forall\, x_i, x_j$ in the vertex set. The interesting part about this graph, however, is that all of the arcs seem to branch out of a central hub, c, which has maximum in-degree and out-degree.

5 Conclusion

The subject of primitive matrices and graphs has kept me engaged throughout my summer scholarship. I believe that there is a lot more research to be done on these topics, especially when you view them under finite fields. You could spend hours sifting through these primitive matrices and their links to the graphs generated from them. I am extremely thankful for the opportunity to conduct such a research project, and I was satisfied with the results achieved.

References

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