# Using Bäcklund Transformations to Find Soliton Solutions

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July 27, 2022

**Abstract**

Bäcklund transformations are a useful method to solve for soliton solutions. In this paper, Bäcklund transformations are analysed and used to solve the one and two-soliton solutions for the sine-Gordon equation and the Korteweg-De Vries equation.

## 1 Introduction

Bäcklund transformations (BTs) were developed in the 1880s by L.Bianchi and A.V.Bäcklund and were originally used in the subject of differential geometry. BTs transform a nonlinear partial differential equation (PDE) into another PDE. They relate PDEs and their solutions and they are used to generate solutions to nonlinear PDEs. They typically consist of a system of two first order PDEs relating two functions. If the two functions satisfy the PDE separately, then the two functions is a BT. If both functions satisfy the same PDE then it is an auto-Bäcklund transformation [1].

In this paper, soliton solutions to the sine-Gordon (SG) equation and to the Korteweg-De-Vries (KdV) equation are analysed. Solitons were first observed by John Scott Russell in 1834. He described that he was watching a boat on the Edinburgh-Glasgow canal when it suddenly came to a stop. The water which had been put in motion accumulated around the boat and rolled forward rapidly. The wave had a rounded, smooth shape and it continued moving forward without changing form or slowing down. Scott Russell described this phenomenon as the "Wave of Translation".

Solitons are described as waves with just a single crest that maintain their shape while they propagate at a constant velocity. Scott Russell found that their speed depend on the size of the wave and their width compared to the depth of the water. Solitons never merge but instead a small wave can be overtaken by another bigger one which results in a phase shift. If a wave is too big for the depth of the water it splits into two, one larger than the other. Another interesting aspect of solitons is the way in which they are moving. Travelling solitons are either kink or antikink depending on whether they are propagating clockwise or anticlockwise respectively. Collisions between kink-kink solitons behave differently than between kink-antikink solitons. This fact is demonstrated in section 2.2.2 for the two-soliton solution to the SG equation [2].

## 2 Sine-Gordon Equation

Some of the earliest BTs were for the sine-Gordon (SG) equation which is the classical wave equation with a nonlinear sine source term and takes the form:

$$u_{xt} = \sin(u) \tag{1}$$

At first, this equation was considered in differential geometry to describe pseudospherical surfaces. The SG equation is known to be a model for various wave phenomena including the propagation of dislocations in crystals, waves along lipid membranes and torsion waves in strings and pendulums [3]. The interesting aspect of the solutions of this equation comes from the fact that they exhibit particle like solutions.

### 2.1 Bäcklund transformation of the sine-Gordon equation

The two equations:

$$\frac{1}{2}(u+v)_x = a\sin\left(\frac{u-v}{2}\right) \qquad \frac{1}{2}(u-v)_t = \frac{1}{a}\sin\left(\frac{u+v}{2}\right) \tag{2}$$

form the Bäcklund transformation for the sine-Gordon eq. (1) [4]. This is seen by forming the cross-derivatives of each equation:

$$\frac{1}{2}(u+v)_{xt} = \frac{a}{2}(u-v)_t\cos\left(\frac{u-v}{2}\right) \qquad \frac{1}{2}(u-v)_{xt} = \frac{1}{2a}(u+v)_x\cos\left(\frac{u+v}{2}\right) \tag{3}$$

By adding and subtracting these equations respectively, we can conclude that:

$$u_{xt} = \sin(u) \quad \text{and} \quad v_{xt} = \sin(v) \tag{4}$$

Clearly, both $u$ and $v$ satisfy the SG equation. Therefore, using the definition of an auto-Bäcklund transformation which was explained previously, the pair of equations eq. (2) are an auto-Bäcklund transformation for the SG equation.

### 2.2 Constructing the one and two-soliton solution to the SG equation

#### 2.2.1 One-soliton solution

By analysing eq. (4), it can be seen that they both have the zero solution, namely $u(x,t) = 0$ and $v(x,t) = 0$ respectively for all $x$ and $t$. By choosing one of these variables equal to zero, say $v = 0$, the one-soliton solution can be found [5]. Thus, eq. (2) becomes:

$$\frac{1}{2}u_x = a\sin\left(\frac{u}{2}\right) \qquad \frac{1}{2}u_t = \frac{1}{a}\sin\left(\frac{u}{2}\right) \tag{5}$$

To solve for $u$, these equations are integrated:

$$\frac{1}{2}\int^u \frac{du}{\sin\left(\frac{u}{2}\right)} = a\int^x dx \qquad \frac{1}{2}\int^u \frac{du}{\sin\left(\frac{u}{2}\right)} = \frac{1}{a}\int^t dt \tag{6}$$

After integration, the equations become:

$$\log\left|\tan\left(\frac{u}{4}\right)\right| + f(t) = ax \qquad \log\left|\tan\left(\frac{u}{4}\right)\right| + g(x) = \frac{t}{a} \tag{7}$$

where f and g are arbitrary functions of integration.

By subtracting and rearranging eq. (7), the following solution can be constructed:

$$f(t) - ax = g(x) - \frac{t}{a} \tag{8}$$

It then follows that:

$$f(t) = K - \frac{t}{a} \qquad\qquad g(x) = K - ax \tag{9}$$

where $K$ is an arbitrary constant.

Hence, eq. (7) can now be written as the single equation:

$$\log\left|\tan\left(\frac{u}{4}\right)\right| + K - \frac{t}{a} = ax \tag{10}$$

Rearranging to solve for $u$ results in the equation:

$$u(x,t) = 4\arctan\left[C\exp\left(ax + \frac{t}{a}\right)\right] \tag{11}$$

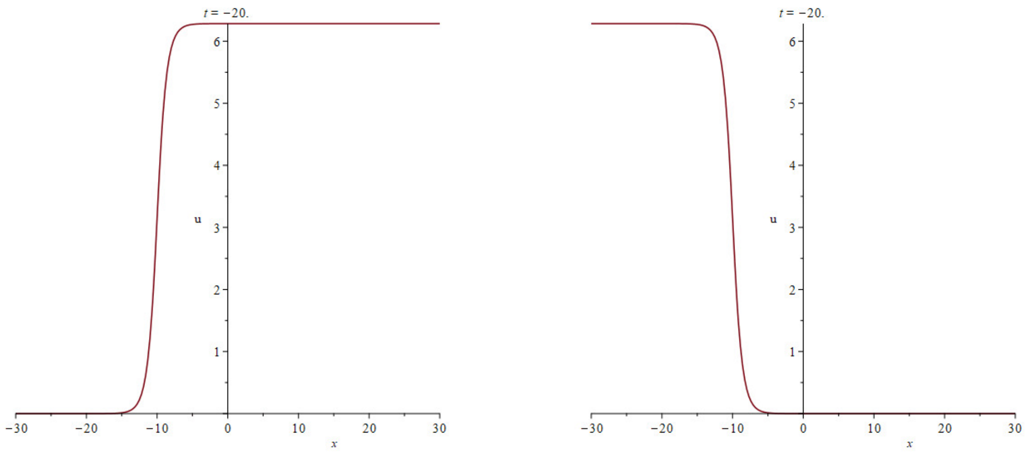
where $C = exp(-K)$.

Eq. (11) is the single-soliton solution. Since a soliton is a wave it can be written in the same form as a wave function, namely:

$$u(x,t) = 4\arctan\left[\exp\left(\frac{x - vt + x_0}{\sqrt{1-v^2}}\right)\right] \tag{12}$$

where $0 < v < 1$ controls the speed of the wave and $x_0$ sets the initial position of the wave.

To be more precise, eq. (12) is the kink one-soliton solution to the SG equation. To find the antikink one-soliton solution, the exponential is raised to the negative power. The two animations below show both a kink and antikink single-soliton solution to the SG equation. The graphs were made in the mathematical software *Maple*.




(a) Kink soliton

(b) Antikink soliton

Figure 1: Short animations showing the one-soliton solutions to the SG equation

As it was explained earlier, kink solitons propagate clockwise while antikink solitons propagate anticlockwise. This can be seen in the graphs above as the graphs are “opposites” of each other.

### 2.2.2 Two-soliton solution

The two-soliton solution to the SG equation can be calculated using the one-soliton solution and a nonlinear superposition. This is possible due to Bianchi's Permutability Theorem (BPT) which provides a way to obtain new solutions. It states that given three distinct solutions to a PDE, a fourth solution can be found algebraically using a suitable auto-Bäcklund transformation.

Say $u_0$ is a solution of a PDE with an auto-Bäcklund transformation. One can apply the BT to $u_0$ with parameter $a_1$ to obtain the solution $u_1$. Then, one can take the BT from $u_1$ with parameter $a_2$ to get the solution $u_{12}$. Similarly, one can take the BT from $u_0$ with parameter $a_2$ to get $u_2$ and then take another BT from $u_2$ with parameter $u_1$ to obtain the solution $u_{21}$.

BPT shows that the solutions $u_{12}$ and $u_{21}$ are the same. This fact can be used to obtain another solution for the PDE, and in this case, the SG equation. The diagram below helps to illustrate BPT.

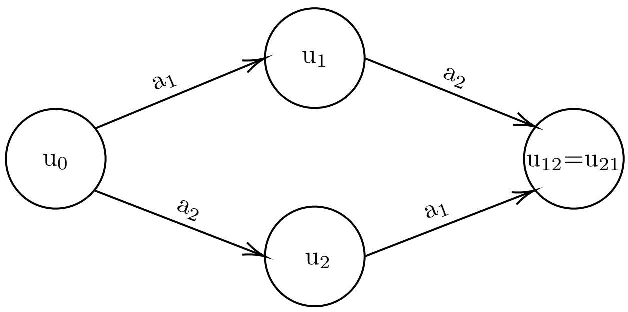



Figure 2: Diagram illustrating BPT

The two-soliton solution can now be calculated using the one-soliton solution and our knowledge from BPT. Firstly, two solutions $u_1$ and $u_2$ are generated from the solution $u_0$ with two different parameters, in this case $a_1$ and $a_2$ respectively [5]:

$$\frac{1}{2}(u_1+u_0)_x = a_1 \sin\left(\frac{u_1-u_0}{2}\right) \qquad \frac{1}{2}(u_2+u_0)_x = a_2 \sin\left(\frac{u_2-u_0}{2}\right) \tag{13}$$

Then, two more solutions are constructed, $u_{12}$ by taking the BT from $u_1$ with parameter $a_2$ and $u_{21}$ by taking the BT from $u_2$ with parameter $a_1$:

$$\frac{1}{2}(u_{12}+u_1)_x = a_2 \sin\left(\frac{u_{12}-u_1}{2}\right) \qquad \frac{1}{2}(u_{21}+u_2)_x = a_1 \sin\left(\frac{u_{21}-u_2}{2}\right) \tag{14}$$

Now BPT tells us that $u_{12} = u_{21}$. Using this fact, the two-soliton solution to the SG can be obtained. This is done by subtracting the difference of eqs. (13) from the difference of eqs. (14) to get:

$$\begin{aligned}&\frac{1}{2}[(u_{12}+u_1)_x - (u_{12}+u_2)_x - (u_1+u_0)_x + (u_2+u_0)_x] = \\ &\qquad a_2 \sin\left(\frac{u_{21}-u_1}{2}\right) - a_1 \sin\left(\frac{u_{21}-u_2}{2}\right) - a_1 \sin\left(\frac{u_1-u_0}{2}\right) + a_2 \sin\left(\frac{u_2-u_0}{2}\right)\end{aligned} \tag{15}$$

This simplifies into:

$$-a_1\left[\sin\left(\frac{u_{21}-u_2}{2}\right)+\sin\left(\frac{u_1-u_0}{2}\right)\right]+a_2\left[\sin\left(\frac{u_{12}-u_1}{2}\right)+\sin\left(\frac{u_2-u_0}{2}\right)\right]=0 \tag{16}$$

To find the two-soliton solution, $u_{12}$ has to be solved for. The easiest way to do this is to first use the trigonometric identity: $\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$ The equation thus becomes:

$$\begin{aligned}-a_1\left[\sin\left(\frac{u_{12}-u_2+u_1-u_0}{4}\right)\cos\left(\frac{u_{12}-u_2-u_1+u_0}{4}\right)\right]+&\\ a_2\left[\sin\left(\frac{u_{12}-u_1+u_2-u_0}{4}\right)\cos\left(\frac{u_{12}-u_1-u_2+u_0}{4}\right)\right]&=0\end{aligned} \tag{17}$$

By removing the common cosine factor, the equation is simplified to:

$$-a_1\sin\left(\frac{u_{12}-u_2+u_1-u_0}{4}\right)+a_2\sin\left(\frac{u_{12}-u_1+u_2-u_0}{4}\right)=0 \tag{18}$$

Next, the trigonometric identity $\sin(A+B)=\sin A\cos B+\cos A\sin B$ is applied:

$$\begin{aligned}-a_1\left[\sin\left(\frac{u_{12}-u_0}{4}\right)\cos\left(\frac{u_1-u_2}{4}\right)+\cos\left(\frac{u_{12}-u_0}{4}\right)\sin\left(\frac{u_1-u_2}{4}\right)\right]+&\\ -a_2\left[\sin\left(\frac{u_{12}-u_0}{4}\right)\cos\left(\frac{u_2-u_1}{4}\right)+\cos\left(\frac{u_{12}-u_0}{4}\right)\sin\left(\frac{u_2-u_1}{4}\right)\right]&=0\end{aligned} \tag{19}$$

By grouping like terms and using the fact that $\cos(-A)=\cos A$ and $\sin(-A)=-\sin A$, the equation can be simplified to:

$$\sin\left(\frac{u_{12}-u_0}{4}\right)\cos\left(\frac{u_1-u_2}{4}\right)(-a_1-a_2)+\cos\left(\frac{u_{12}-u_0}{4}\right)\sin\left(\frac{u_1-u_2}{4}\right)(-a_1+a_2)=0 \tag{20}$$

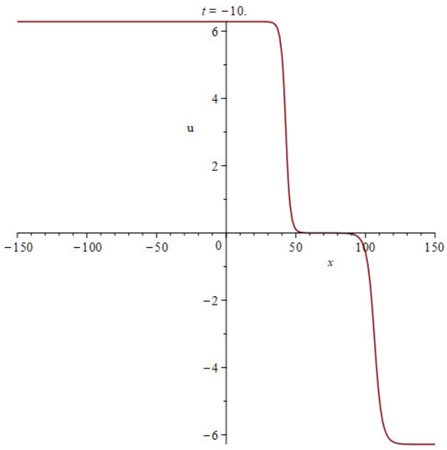
Since, $\tan A=\frac{\sin A}{\cos A}$, dividing the equation by $\cos\left(\frac{u_{12}-u_0}{4}\right)\cos\left(\frac{u_1-u_2}{4}\right)$ results in:

$$\tan\left(\frac{u_{12}-u_0}{4}\right)(-a_1-a_2)+\tan\left(\frac{u_1-u_2}{4}\right)(-a_1+a_2)=0 \tag{21}$$

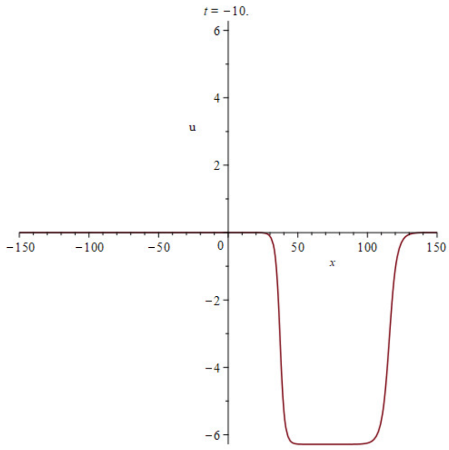
Finally, from this expression, $u_{12}$ can be recovered:

$$u_{12}=u_0-4\arctan\left[\left(\frac{a_1+a_2}{a_1-a_2}\right)\tan\left(\frac{u_1-u_2}{4}\right)\right] \tag{22}$$

Eq. (22) can be used to find a fourth solution to the SG equation given that three solutions are already known. By letting $u=0$ and by letting $u_1$ and $u_2$ be non-zero one-soliton solutions, the two-soliton solution can be found algebraically. The two animations below show the kink-kink and antikink-kink solutions respectively.




(a) Kink-kink soliton




(b) Antikink-kink soliton

Figure 3: Short animations showing the two-soliton solutions to the SG equation using BPT

In both collisions the waves pass through each other. The wave which starts furthest to the right is faster than the other wave. It catches up to the slower wave until they collide at $t = 0$. As these waves are solitons, the faster wave passes through the other wave instead of merging together. The waves maintain their velocity and shape after the collision and only observe a phase shift. Therefore, these collisions are elastic.

Another two-soliton solution called breathers can also arise. Breathers are nonlinear waves where energy is concentrated in a localized area. Breathers experience oscillatory behaviour [6]. Below is an animation of a standing breather which is a coupled kink-antikink soliton.

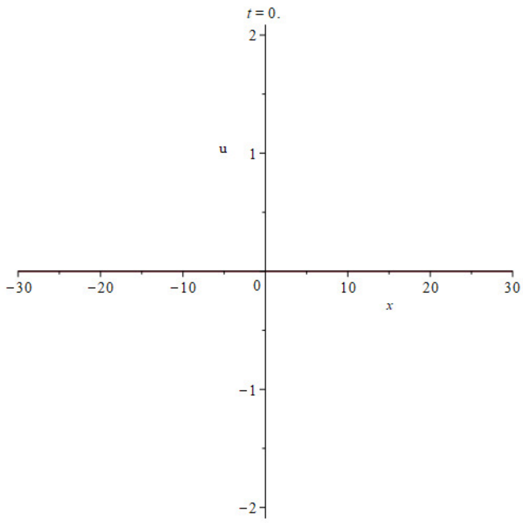



Figure 4: Animation showing a standing breather solution to the SG equation

Small amplitude breathers can also arise. The animations below shows this type of breather. This wave has a breather envelope which is also included.

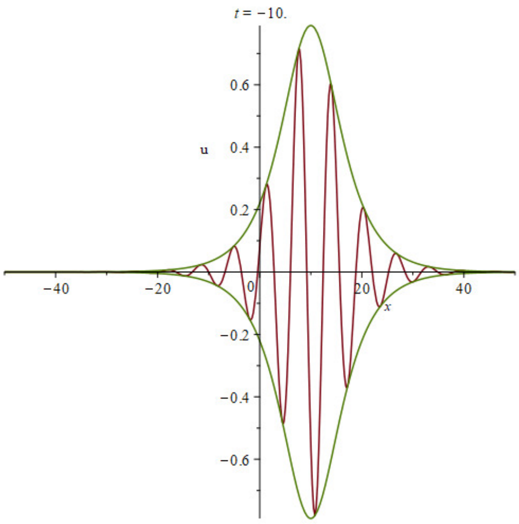



Figure 5: Animation showing a small amplitude breather and its envelope

# 3 Korteweg-De-Vries equation

The Korteweg-De-Vries (KdV) equation is a model for waves on shallow water surfaces and is defined as:

$$u_t - 6uu_x + u_{xxx} = 0 \tag{23}$$

This equation has many applications including long internal waves in a density-stratified ocean, ion acoustic waves in plasma and acoustic waves on a crystal lattice [7].

## 3.1 Constructing the one and two-soliton solution to the KdV equation

### 3.1.1 One-soliton solution

Constructing the one-soliton solution for the KdV equation is slightly more complicated than for the SG equation. Firstly, the Miura transformation has to be used which is the transformation between the KdV equation and the modified KdV equation and is defined as [4]:

$$u = v^2 + v_x \tag{24}$$

Secondly, since the KdV equation is Galilean invariant (that is, the laws of motion are the same in all inertial frames), it is easier to work with $u - \lambda$ rather than just $u$. Therefore, the Miura transformation becomes:

$$u = \lambda + v^2 + v_x \tag{25}$$

where $u = u(x,t)$, $v = v(x,t)$ and $\lambda$ is a real parameter. Hence, the modified KdV equation becomes:

$$v_t - 6(v^2 + \lambda)v_x + v_{xxx} = 0 \tag{26}$$

From eq. (26), if $v$ is a solution then $-v$ is also a solution since the equation is odd in $v$. Two functions can then be introduced:

$$u_1 = \lambda + v^2 + v_x \qquad\qquad u_2 = \lambda + v^2 - v_x \tag{27}$$

where $v$ and $\lambda$ are given. By using simple algebra it is clear that eq. (27) can be written as:

$$u_1 - u_2 = 2v_x \qquad\qquad u_1 + u_2 = 2(\lambda + v^2) \tag{28}$$

For convenience purposes, another transformation is introduced:

$$u_i = \frac{\partial w_i}{\partial x} \tag{29}$$

where $w = w(x,t)$ and $i = 1, 2$ By using eq. (29) and following a couple calculations eq. (28), can be written as:

$$w_1 - w_2 = 2v \qquad\qquad (w_1 + w_2)_x = 2\lambda + \frac{1}{2}(w_1 - w_2)^2 \tag{30}$$

The second part of eq. (30) is the $x$ part of the BT. The $t$ part of the BT can be found by rewriting eq. (26) using the previous equations, namely, eq. (28), eq. (29) and the first part of eq. (30):

$$(w_1 + w_2)_t - 3(w_{1x}^2 - w_{2x}^2) + (w_1 - w_2)_{xxx} = 0 \tag{31}$$

The BT of the KdV is made up of equations eq. (30) and eq. (31) which actually make up an auto-Bäcklund transformation. These equations along with eq. (29) are used to find solutions to the KdV.

Similar to the SG equation, the one-soliton solution for the KdV is found by letting $w_2 = 0$ for all $x$ and $t$. By following the same method as before, the BT yields:

$$w_1 = -2k \tanh(k(x - x_0 - 4k^2 t)) \tag{32}$$

for $|w_1| < 2k$ where $\lambda = -k^2$. This is the one-soliton solution for the modified KdV equation. Hence, using eq. (29), the one-soliton solution of the actual KdV equation can be found for this particular condition:

$$u_1 = -2k^2 \operatorname{sech}^2(k(x - x_0 - 4k^2 t)) \tag{33}$$

However, for $|w_1| > 2k$, the Bäcklund transformation yields:

$$w_1 = -2k \coth(k(x - x_0 - 4k^2 t)) \tag{34}$$

and so the single-soliton solution for the KdV equation when $|w_1| > 2k$ is:

$$u_1 = -2k^2(1 - \coth^2(k(x - x_0 - 4k^2 t)) \tag{35}$$

The short animation below illustrates a one-soliton KdV solution with $|w_1| < 2k$.

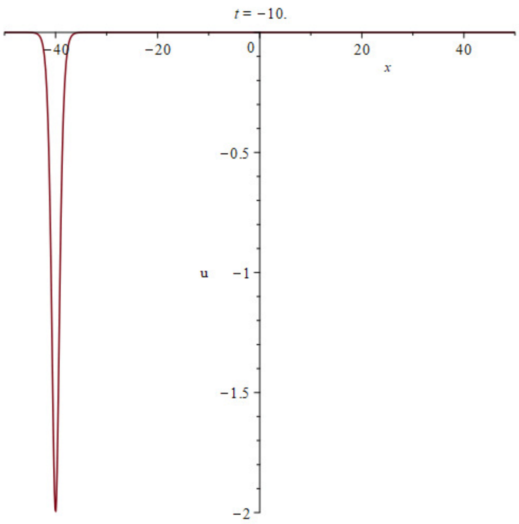



Figure 6: Animation showing an example of a one-soliton solution to the KdV equation

### 3.1.2 Two-soliton solution

The method to go from a one-soliton solution to a two-soliton solution was explained for the SG equation. The same method is used to obtain the two-soliton solution for the KdV, that is, by using the one-soliton solution and BPT.

Following calculations, the two-soliton solution to the modified KdV is found to be:

$$w_{12} = w_0 - \frac{4(\lambda_1 - \lambda_2)}{w_1 - w_2} \tag{36}$$

This equation indicates that if three solutions $w_0$, $w_1$ and $w_2$ to the modified KdV equation are given, then a fourth solution $w_{12}$ can be derived. Eq. (29) can then be used to find the solution to the KdV equation. For example, for a two-soliton solution, one can take $w_0 = 0$, $w_1 = -2\tanh(x - 4t)$ and $w_2 = -6\coth(3x - 108t)$, which are three single-soliton solutions to the modified KdV equation Eq. (36) then becomes:

$$w_{12} = \frac{-32}{6\coth(3x - 108t) - 2\tanh(x - 4t)} \tag{37}$$

Now, to obtain the two-soliton solution for these values of $w_0$, $w_1$ and $w_2$, eq. (29) is used and so eq. (37) needs to be integrated. Adding an integration constant simply shifts the graph on the vertical axis and so it is set to zero. Using the quotient rule, the integration yields:

$$u_{12} = 64\left[\frac{9(1 - \coth^2(3x - 108t) - \operatorname{sech}^2(x - 4t))}{\{6\coth(3x - 108t) - 2\tanh(x - 4t)\}^2}\right] \tag{38}$$

The animation below shows how this equation behaves as time is varied. As it can be seen there are two waves: one bigger than the other. The bigger wave travels faster and overtakes the smaller wave. This shows that Scott Russell's observations were correct. Solitons don't merge but instead the larger, faster wave overtakes the smaller wave.

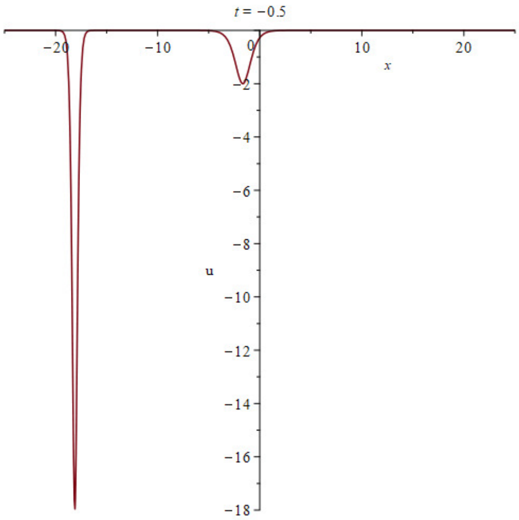



Figure 7: Animation showing a two-soliton solution to the KdV equation

The waves in the animation above maintain their speed and shape as is expected for solitons. However, the collision creates a phase shift. This is easily seen by animating the one-soliton solution on top of the two-soliton solution.

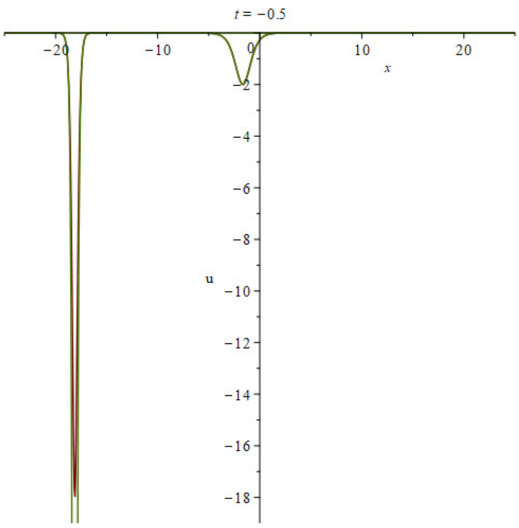



Figure 8: Animation showing the phase shift following the collision of the two-soliton solution to the KdV equation

In the animation above, the wave represented by the red line is the two-soliton solution and the waves represented by the green lines are two separate one-soliton solutions. The faster wave is shifted forward while the smaller, slower wave is shifted backwards. Both waves are not shifted by the same amount, namely, in this example, the faster wave is shifted less than the slower wave.

# 4 Conclusion

In conclusion, the one-soliton solutions to the SG and KdV equations were found using the method of BTs. BPT was then introduced to find the general formula for the two-soliton solution for both equations. The animations created in *Maple* illustrated various solutions to these equations such as the one and two-soliton solutions and even breathers for the SG equation. These animations showed the various properties of solitons, namely that their speed depend on their size and that they maintain their shape when travelling. Towards the end of the report, it was seen that solitons don't merge but instead pass through each other creating a phase shift, similar to what Scott Russell observed in 1834.

Obviously, more research can be done in the area of BTs and solitons. Firstly, no method has been found yet on how to generate BTs consistently and therefore more work can be done on this subject. Secondly, more can be researched on solitons in both the SG and KdV equations. The one and two-soliton solutions on top of the knowledge gathered through BPT can be used to form three-soliton solutions or even n-soliton solutions. The phase shift created when one soliton passes through another one can be calculated for both the SG and KdV equations. The idea of solitons can also be extended to two dimensions which are then called vortices.

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